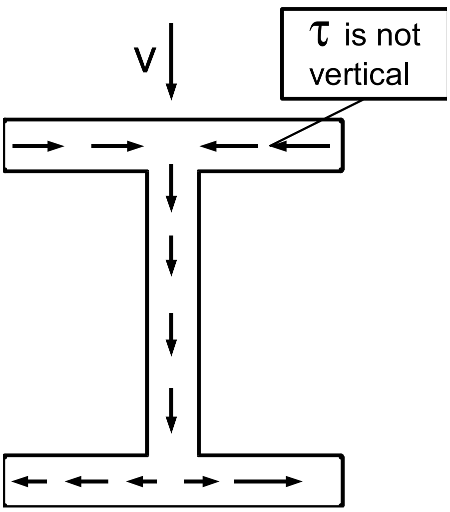
Table of Contents

Transverse Shear of Thin-Walled Beams[1]

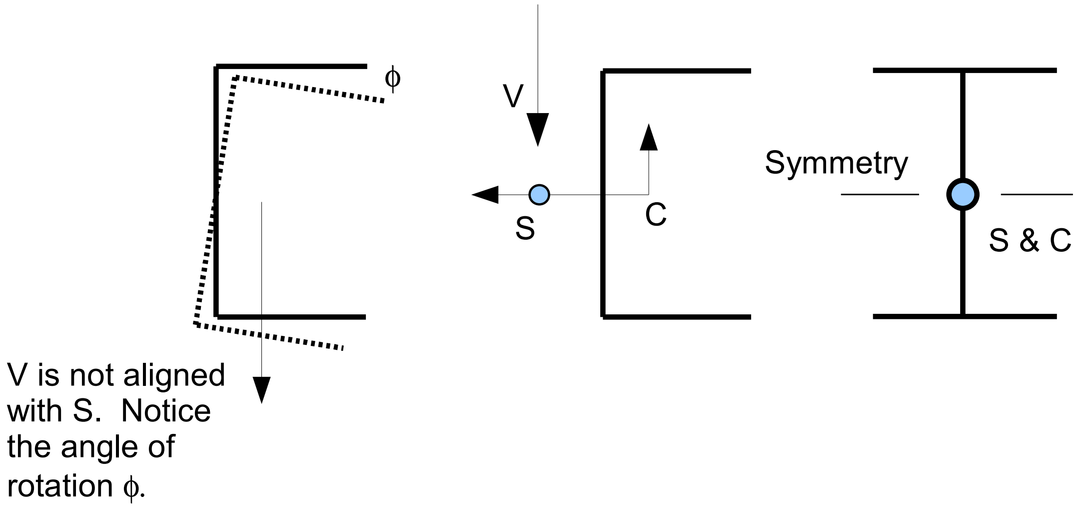
1 Introduction

Beams are subjected to shear stresses given by $\tau = \frac{VQ_z}{I_z t}$. A major difference between a thick and a thin wall cross-section, is that the shear stresses for thin-walled beams are always aligned with the median line of the cross-section, see the figure below.



1 Ahmad Mansour

The shearing force V causes pure bending without twisting moment, only when its line of action passes through a specific point named the shear center (S). The shear centers for two cross-sections are shown below (the centroids C are also depicted to show that S and C are in general two different points):



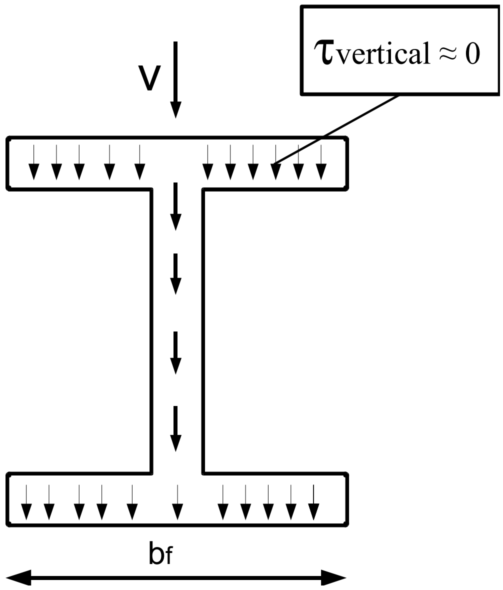
For an I beam, the shear center is at the intersection of the vertical and horizontal axes of symmetry (why?). On the other hand, the shear center of the channel section, lies along the horizontal axis of symmetry. We should apply V by an attachment (which extends outside the borders of the channel), in order to ensure that V passes through the shear center. Otherwise, the force V will cause the cross section to twist by an angle ϕ, as shown above.

2 Objective

1. To prove that τ is aligned with the median line.
2. To determine the location of the shear center.
3. To evaluate τ for situations involving shear stresses due to simultaneous transverse forces and twisting moments.

3 Proof that the Shear Stresses are Aligned with the Median Line

The vertical component of the shear stress in the flanges of the shown wide I beam is negligible because the thickness t in $\tau = \frac{VQ_z}{I_z t}$, equals b_f which is large.



On the other hand, there is a significant component of τ along the median line. We will study the equilibrium of the volume aba'b' (refer to figures a-c below). Where, the longitudinal surface aa' is an external surface free from stresses. While surface bb' is an internal surface transmitting internal forces.

In figure (b):

$$M_2 = M_1 + \frac{dM_1}{dx} dx = M_1 + V dx$$

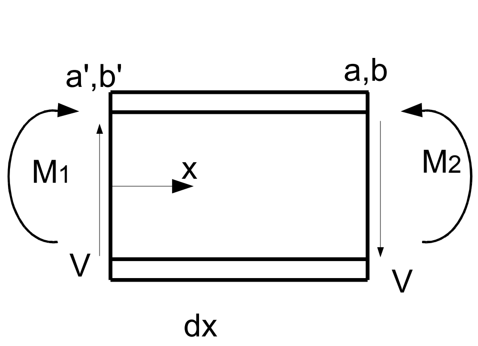
$$|\sigma| = \frac{My}{I_z}$$

Figure (c) shows the forces acting on the free body aba'b'. Faces ab and a'b' each has an area of A^o and a first moment of area Q^o. The forces F_1 and F_2 acting on faces a'b' and ab equal:

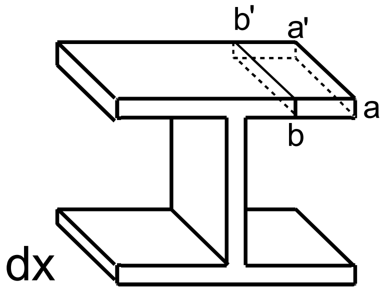
$$F_1 = \int_{a'b'} \sigma \, dA = \int_{A^o} \frac{My}{I_z} dA = \frac{M_1}{I_z} Q^o$$

Similarly

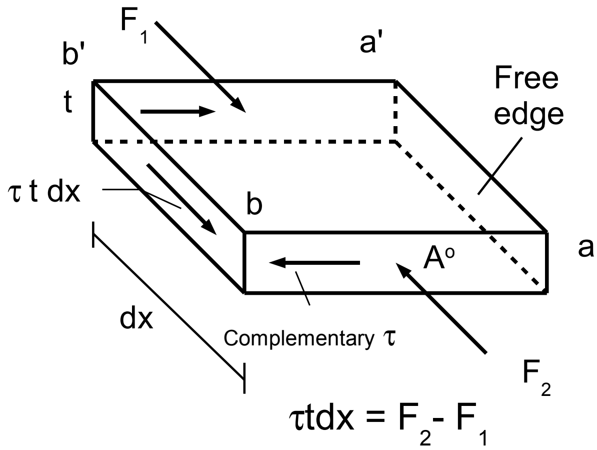
$$F_2 = \int_{ab} \sigma \, dA = \frac{M_2}{I_z} Q^o$$



(b) Side view



(a) A portion of the beam of length dx



(c) A free body diagram abbá

Since, $\sum F_x = 0$ then

$$\tau\, t\, dx = F_2 - F_1 = \frac{M_2 - M_1}{I_z} Q^o = \frac{V}{I_z} Q^o\, dx$$

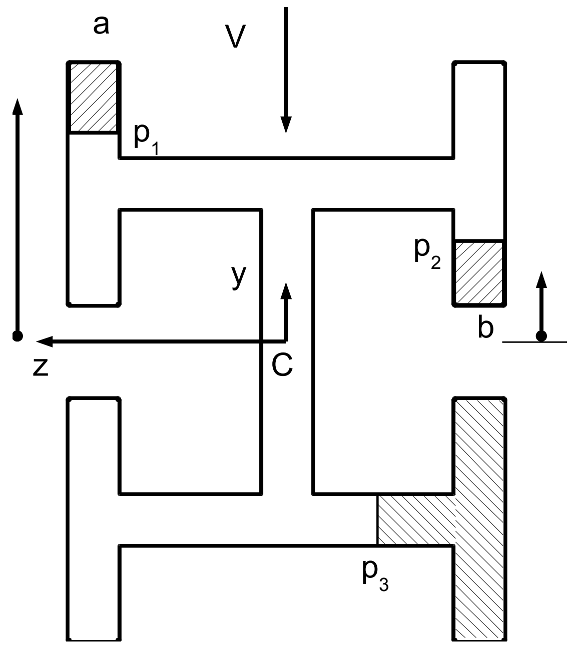
Hence,

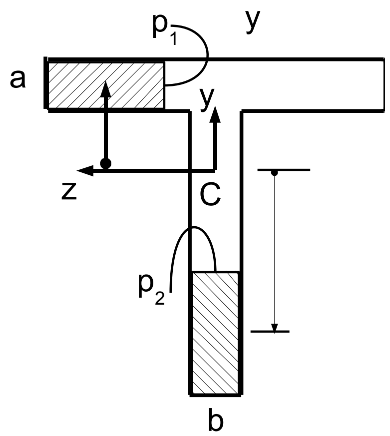
$$q = \tau\, t = \frac{VQ^o}{I_z} \text{ , and the shear stress } \quad \tau = \frac{VQ^o}{I_z t}$$

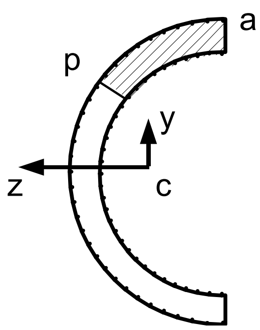
Therefore, the complementary shear stress acting along the plane of the cross-section is aligned with the median line, figure (c).

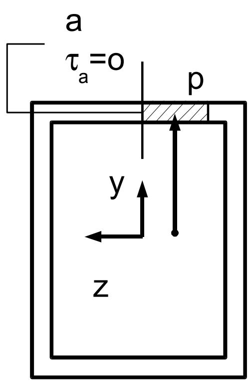
4 Samples of Relevant Areas in Calculating the Shear Stress

The formula $\tau = \dfrac{VQ^o}{I_z t}$ is valid for any thin-walled cross-section. The first moment of area Q^o is of an area bounded by free external edges from all sides and the internal surface at the generic point p where τ is sought. For instance, the following figures show the relevant areas for calculating the shear stress τ









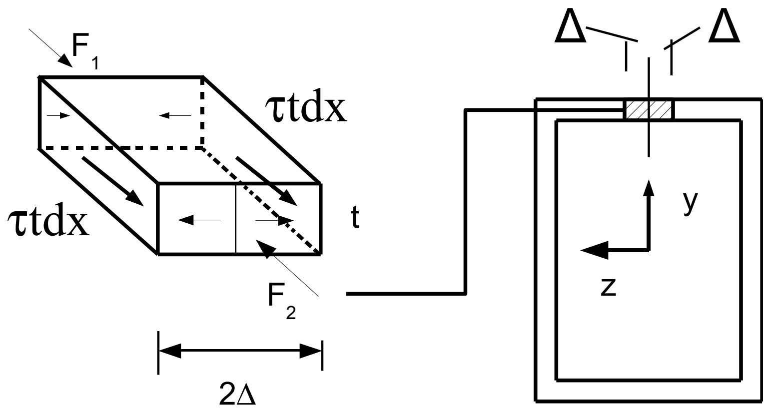
Note: Due to Symmetry
τ = 0 along the y-axis.

The shear stress τ at point 'a' of the above rectangular tube will be shown to equal zero. Hence, the internal longitudinal surface passing through 'a' acts like a free edge. In order to prove that $\tau_a = 0$, study the equilibrium of the following free body diagram

Since, $\Sigma F_x = 0$ then

$2\ \tau\ \tau t dx = F_2 - F_1 = (Q/I_z)\ V\ dx = 0$

Where Q = 0, because the area taken from the cross section is infinitesimally small. From now on, we can use the symbol Q instead of Q^0.



τ = o along the axis of symmetry

5 Sign Convention for the Shear Stress τ

The equilibrium equation determines the direction of the shear stress at any point in the cross-section. However, there are two handy methods to estimate the shear stress direction, namely:

5.1 (a) Fluid Analogy

The shear flow q at any junction, behaves like a fluid[2]. For instance at point 'b', the shear flow at the web is downward and emanates from the junction 'b'. Hence, the shear flow q at each side of the flange flows toward the junction 'b'. At junction 'd', the shear flow q along the horizontal segment db flows out from 'd'. Hence, along the vertical segment, the two shear flows are directed toward 'd'.

5.2 (b) Sign Convention

Draw an arrow (inside the area for which Q is to be evaluated) pointing to the point of interest (see points a and b below). The direction of the arrow is the positive direction of τ in the formula $\tau = \frac{VQ}{I_z t}$. Where,

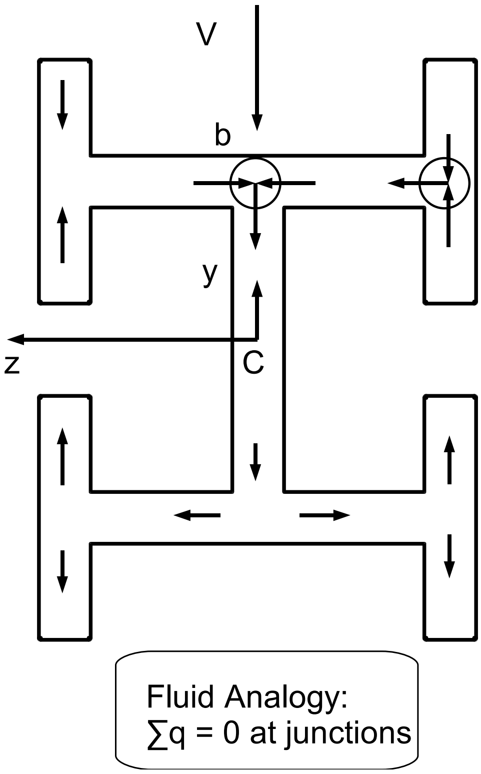
V is positive when it is downward.

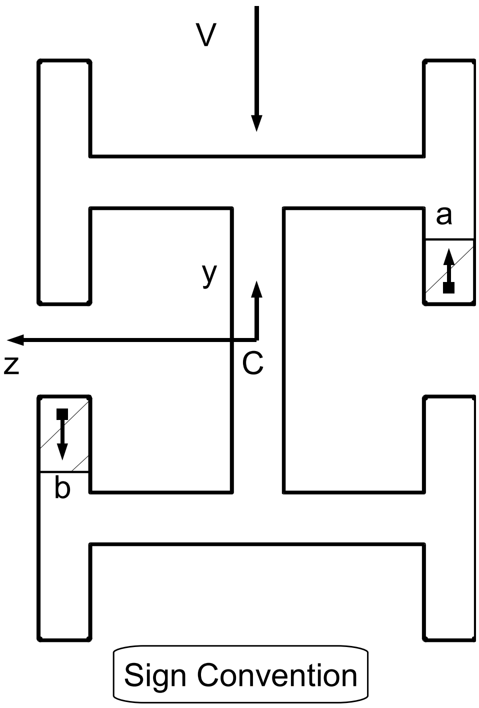
Q is positive or negative depending on the location of its center of area. For point a, Q is positive, and for b, Q is negative.

The shear formula gives a positive value for τ_a and a negative value for τ_b. Therefore, τ_a is upward

2 Σ q = 0 at any junction. Try to prove it. The proof is similar to the previous proof of τ = 0 at at the axis of symmetry of a tube.

in the direction of the arrow. Since, τ_b is negative, its direction is opposite to that of the arrow. Therefore, τ_b flows in the upward direction.



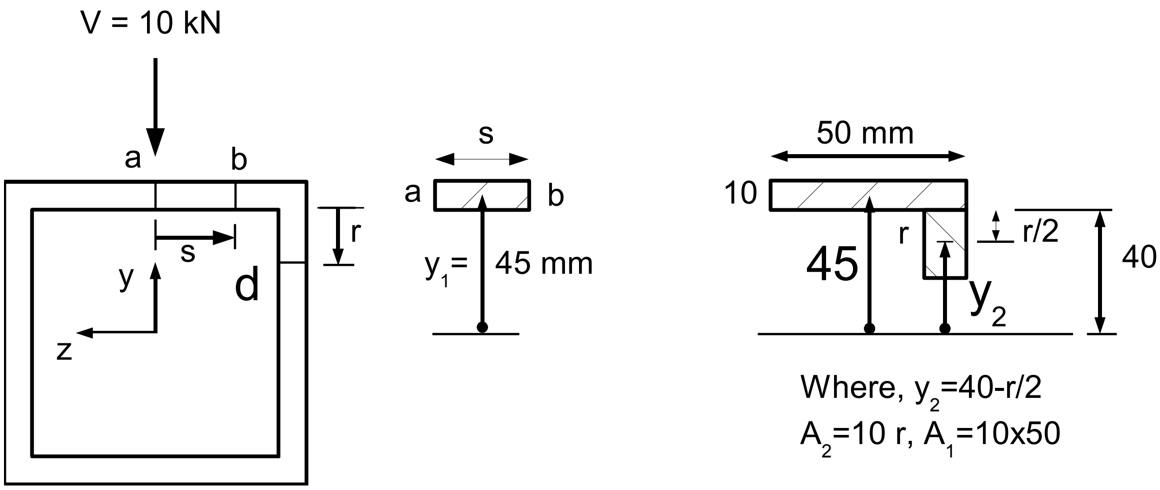


6 Response of Beams when V is Aligned with an Axis of Symmetry

This ensures that the beam is free from torsion. The following examples of sections 6.1 and 6.2 show how to calculate the shear stress distribution.

6.1 Example

Derive the expressions of the shear stress at point b and d in terms of the contour coordinates s and r, refer to figure.



Solution:

$I_z = I_y = \frac{1}{12}\ (100^4 - 80^4) = 4\,920\,000\ \ mm^4$

Shear stress along a-b.

$Q_{z\ at\ b} = (t\ s)\ 45 = 450\ s\ \ mm^3$

$$\tau_b = \frac{V\,Q_{z\,at\,b}}{I_z\,t} = \frac{(10000)(450\,s)}{(4920000)(10)} = 0.0914634\,s\ \ \text{MPa}$$

The shear stress varies linearly with s. At s = 40 mm τ = 3.66 MPa.

Shear stress along the vertical segment.

$Q_{z\ at\ d} = A_1\ y_1 + A_2\ y_2 = (50)\ (10)\ (45) + 10\ r\ (40 - 0.5\ r) = 22500 + 10\ r\ (40 - 0.5\ r).\ \ mm^3.$

$$\tau_{z\,at\,d} = \frac{V\,Q_z}{I_z\,t} = \frac{1}{4920}(22500 + 400\,r - 5\,r^2)$$

The shear stress is maximum at the y = 0. We can check that by:

$$\frac{d\,\tau}{d\,r} = \frac{1}{4920}(400 - 10\,r) = 0\ \text{Hence},\ r = 40\ \text{mm}$$

Therefore, the maximum shear stress τ = 6.2 MPa.

6.1.1 Comments

(1) The figure associated with this comment shows the direction and magnitude of the shear stress.

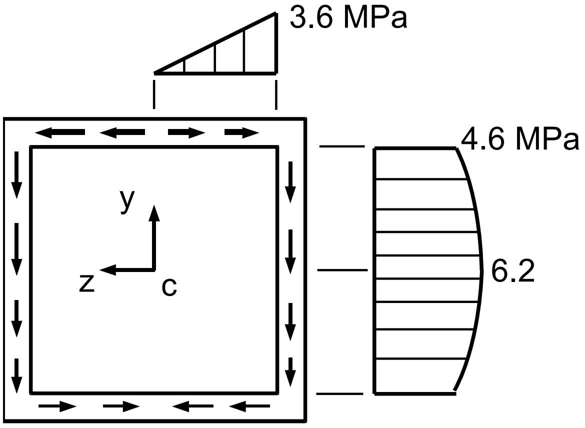
(2) In solving the previous example, we have dealt with the actual dimensions of the cross-section. However, we may use median line representation of the cross-section. We simply draw the median line and assume that each segment of the line has a thickness t, refer to figure. The cross-section in effect is represented by partially overlapping rectangles. Therefore, this introduces a slight inaccuracy in calculating the moment of inertia. On the other hand, this method defines accurately the starting and ending points of each segment.

(3) The applied transverse force (V) is transmitted through the walls of the cross-section as shown below. Where,

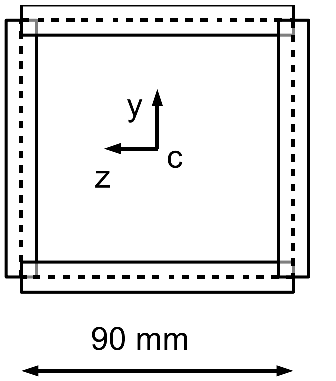
$$F_h = \int_0^{45} \tau\,t\,ds = 0.914634\left[\frac{s^2}{2}\right]_0^{45} = 926\ \text{N}$$

The resultant of these forces is a vertical force V aligned with the axis of symmetry y (as expected). However, for cross-sections without an axis of symmetry, the line of action of the resultant force determines the location of (V) needed to avoid twisting of the cross-section.

Note that we used the median line length (45 mm) in calculating F_h.

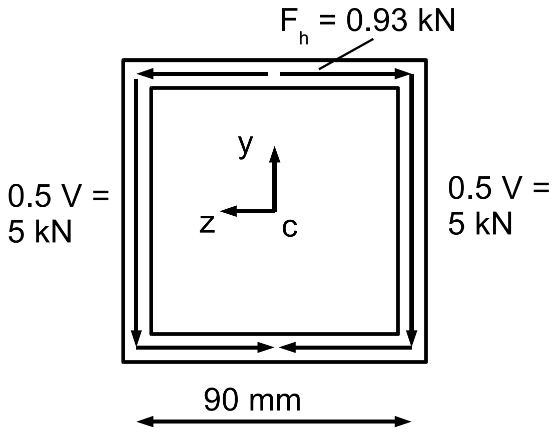


Comment (1)



Median line representation

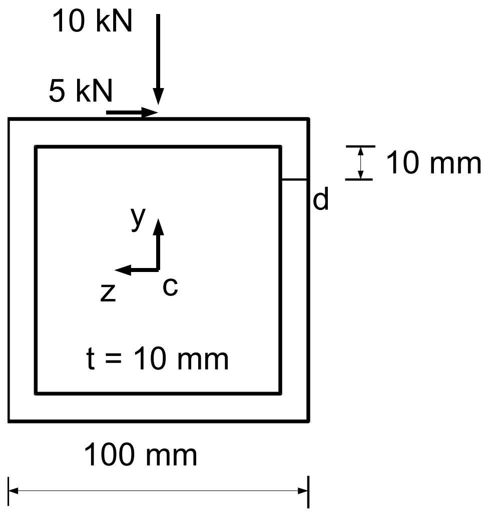
Comment (2)



Comment (3)

6.2 Example

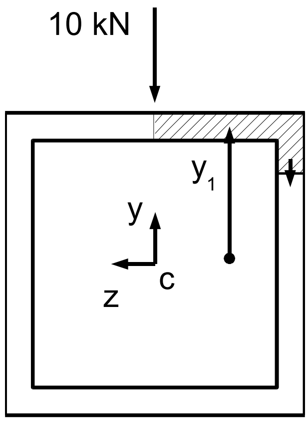
An inclined shearing force is applied to the same square tube of example (1), refer to the figure below. Determine the shear stress and its direction at point d.



Solution:

This inclined force has a vertical component of 10 kN acting along the y-axis of symmetry. The horizontal component of 5 kN is not aligned with the symmetric z-axis. Therefore, replace this force by a force passing through the z-axis and a twisting moment T = (5000)(50) = 250 000 N.mm. Divide the problem into three separate ones as shown below. Calculate the shear stress at point d for each case. Algebraically add the shear stresses of the three cases to get the required shear stress at d.

Case (1) V_y = 10 000 N

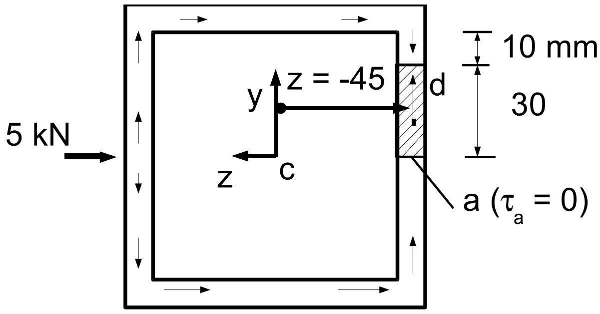


Case (1)
$\tau = (V\ Q_z) / (I_z\ t)$

This case is that of example (1).

τ_d = 5.28 MPa ↓

Case (2) V_z = 5000 N



Q_z of hatched area =
(30x10)(-45) mm^3
The ↑ gives +ve τ direction.
Case (2) $\tau = (5000\ Q_y)/(I_y t)$

$$\tau_d = \frac{V_z Q_y}{I_y t}$$

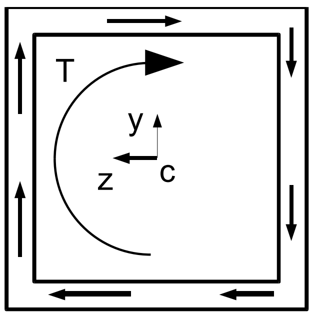
The first moment of area Q_y of the hatched area is:

Q_y = (10 x 30) (-45) = -13 500 mm^3

$$\tau_d = \frac{(5000)(-13500)}{(4920000)(10)} = -1.37 \text{ MPa}$$

The shear stress equals 1.37 MPa ↓, since the positive direction is upward.

Case (3) T = 250 000 N.m



Case (3) $\tau = T/(2tA_0)$

Since, the cross-section is closed then:

$$\tau = \frac{T}{2 t A_0}$$

A_0 = (90) (90) = 8100 mm^2.

Hence,

τ_d = 1.54 MPa ↓

Superposition

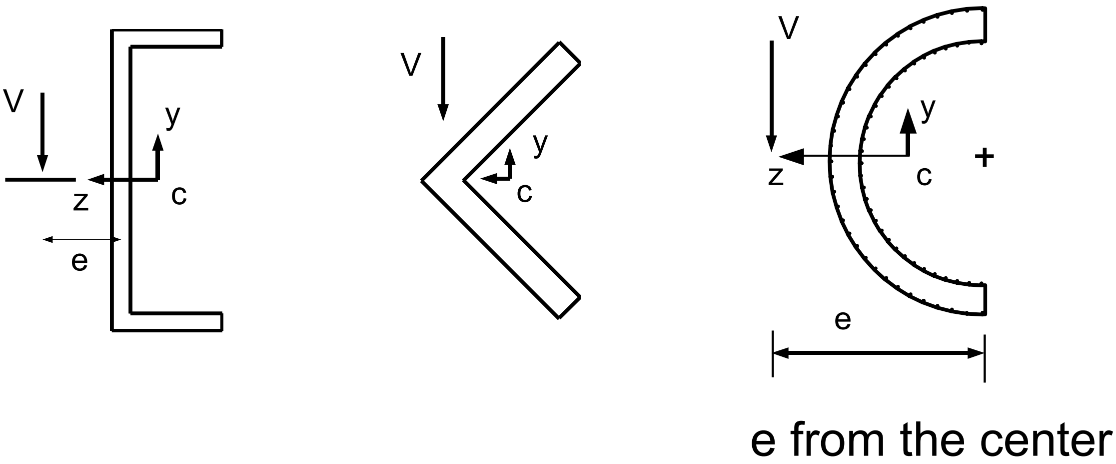
The three shear stresses are pointing downward. That is all of them have the same algebraic sign.

τ_d = 5.28 + 1.37 + 1.54 = 8.2 MPa.

Question: What is the magnitude and location of the maximum shear stress?

7 Beams with V Perpendicular to an Axis of Symmetry – Shear Center

Each of the following cross-sections has the z-axis as an axis of symmetry. For each beam, the applied shearing force is in the y-direction. However, its line of action should be determined such that it will not invoke torsional shear stresses.



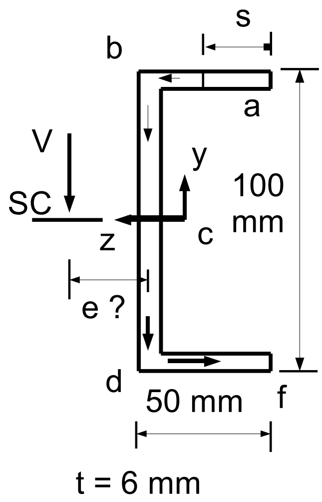
The transverse shear stresses calculated by $\tau = \frac{VQ}{I_z t}$ are associated with forces transmitted through the wall of the cross-section. We can determine (by laws of statics) the line of action of the resultant of these wall forces. This line of action is in itself the line of action of (V). In effect, we determined where (V) should be applied. This ensures that (V) causes pure transverse shear stresses without any torsional effects.

The intersection of the line of action of (V) and the z-axis is the **shear center** (denoted by S or SC in subsequent figures). An inclined shearing force (with V_y and V_z components) passing through the shear center will not induce twisting of the beam.

The following examples will demonstrate how to locate the shear center.

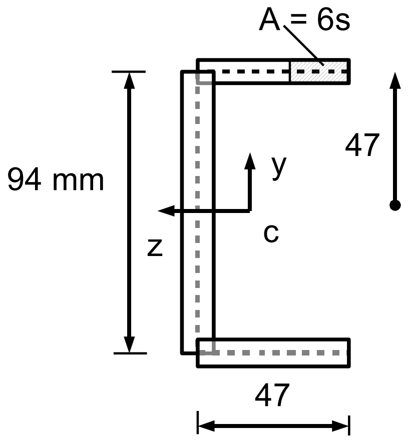
7.1 Example

Determine the location of the shear center (SC).



Solution:

We are going to represent the cross-section by its median line as shown below:



$$I_z = \frac{1}{12}(6)(94^3) + 2\left(\frac{1}{12}(47)(6^3) + (47)(6)(47^2)\right) = 1662860 \quad mm^4$$

Note that, the term involving $t^3 = 6^3$ is very small compared to the other terms. It is usually omitted.

Q_z along the upper flange ab in terms of s.

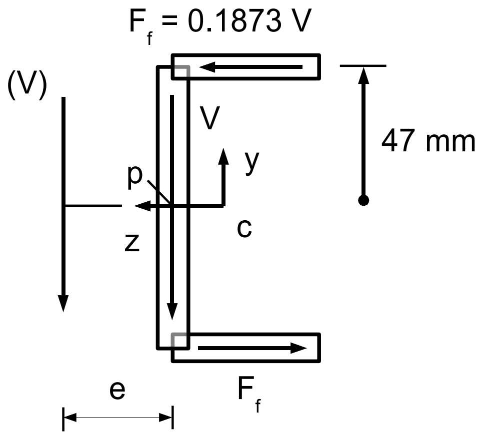
$Q_z = (6\ s)\ 47 = 282\ s \quad mm^3$.

$$\tau = \frac{VQ}{I_z t}$$

The force acting along the upper flange is F_f.

$$F_f = \int_0^{47} \tau\, t\, ds = \int_0^{47} \frac{V\,Q}{I_z t} t\, ds = 0.187309\, V$$

By inspection, the force along the web bd equals V (Why?). In addition, the force acting along the flange de equals F_f, refer to figure.



The figure shows also the resultant (or equivalent) force (V). Since, the resultant is equivalent to the wall forces, its moment about any point in the plane equals to that of the three wall forces.

Take the moment about point p, refer to the previous figure.

(V) e = 0.1873 V x 94 = 17.607 V

Hence,

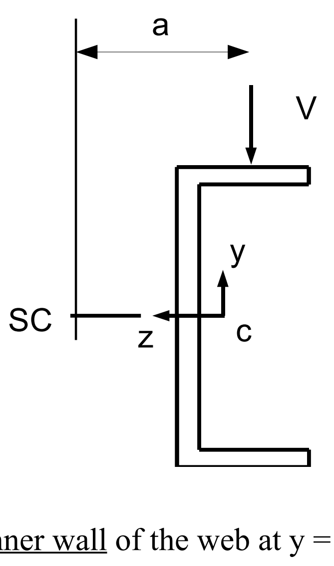
e = 17.6 mm

This is the distance from (V) to the median line of the web. The distance to the left wall of the web is (17.6 - 3) = 14.6 mm.

The shear center is at the intersection of the line of action of (V) and the z-axis. It is a geometric property which is not function of V.

7.1.1 Comments

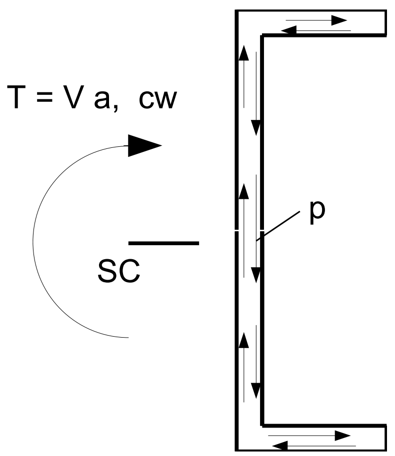
(1) If the transverse force does not pass through the shear center, then the shear stress is composed of transverse and torsional stresses, refer to the figures below.

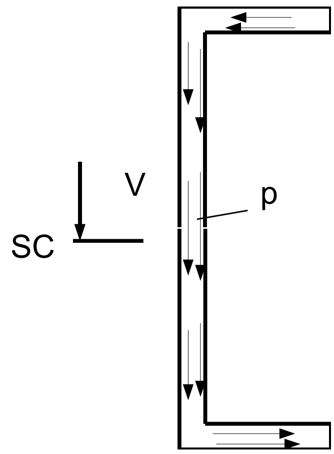


The maximum shear stress is at the <u>inner wall</u> of the web at y = 0 (point p)

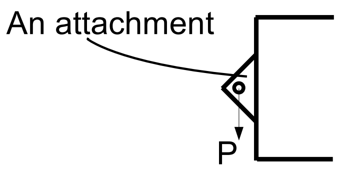
$$\tau_{max} = \frac{V\,Q_{max}}{I_z\,t} + \frac{T}{\frac{1}{3}a\,6^3}$$, where a = (47 + 94 +47) = 188 mm

Note at y=0 and at the outer wall of the web, the stresses should be subtracted.



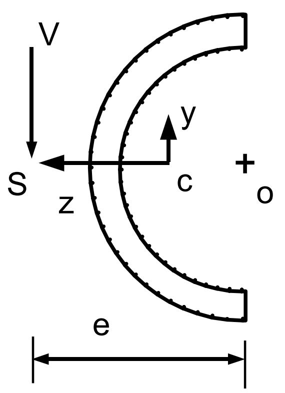


(2) We apply the transverse force at the shear center by using an attachment as shown below:



7.2 Example

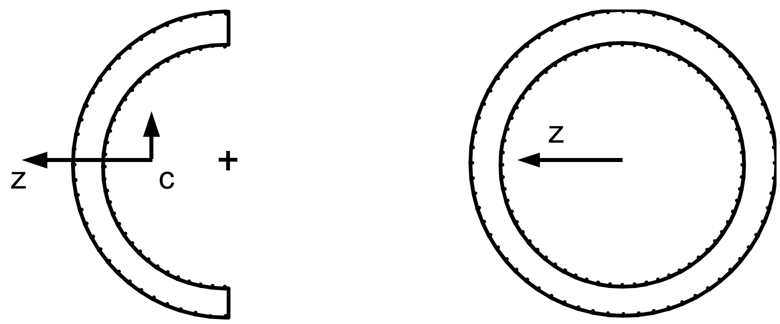
Determine the maximum shear stress and the position of the shear center S (i.e. the distance e) in terms of the mean radius R and the thickness t.



Solution:

$$\tau = \frac{VQ}{I_z t}$$, where

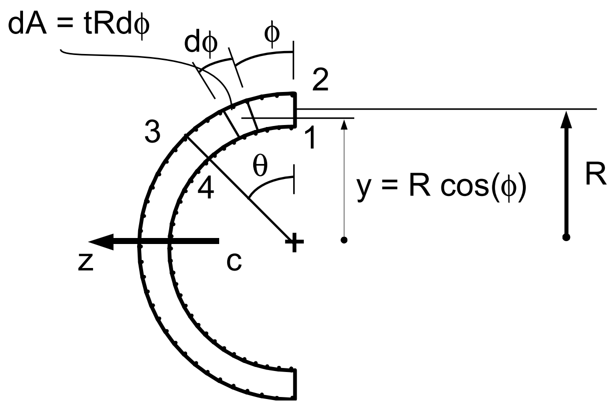
$I_z = 0.5\ \pi\ t\ R^3$ (half the value of a thin tube, refer to the figures below).



$$I_z = \frac{\pi t R^3}{2}$$ $$I_z = \pi\ t\ R^3$$

We want to get τ perpendicular to the radial line 3-4, located at an angle θ from the vertical direction. Therefore, we need to get Q_z for the area 1-2-3-4 using $Q_z = \int y\, dA$. Where, dA is bounded by two radial lines ϕ and $(\phi + d\phi)$.

$dA = t\ R\ d\phi$ with its centroid at $y = R \cos\phi$



Then,

$dQ = y\ dA = t\ r^2 \cos\phi\ d\phi$

$$Q=\int dQ=\int_0^{\theta} t\,R^2\cos\phi\;\;d\,\phi=t\,R^2\sin\theta$$

Therefore, at θ

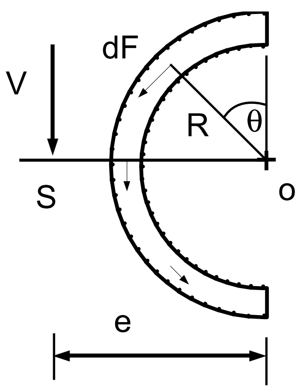
$$\tau=\frac{V\,Q}{I_z\,t}=\frac{V\,t\,R^2\sin\theta}{\frac{\pi\,t\,R^3}{2}\;\;t}=\frac{2\,V}{\pi\,t\,R}\sin\theta$$

At $\theta = 90^o$, τ attains its maximum value,

$$\tau_{max}=\frac{2\,V}{\pi\,Rt}$$

The infinitesimal force at any angle θ is

dF = τ dA = τ t R dθ = $\frac{2}{\pi}V\,d\theta$



These infinitesimal forces transmitted through the wall are equivalent to the original force (V).

Take the moment of forces about O.

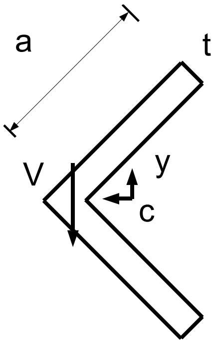
$$(V)e=\int R\,dF=\int_0^{\pi}\frac{2\,V\,R}{\pi}\sin\theta\,d\,\theta=\frac{4\,R\,V}{\pi}$$

Hence,

$$e=\frac{4\,R}{\pi}$$

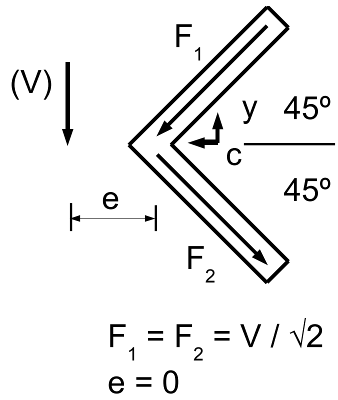
7.3 Example

The equal-leg cross-section has a side length 'a' and a uniform thickness 't', refer to the figure below. Determine the location of the shear center and the shear stress distribution.



Solution:

The shear center is at the corner of the median line. The proof is simple. Assume the transmitted forces along the upper and lower legs to be F_1 and F_2 as shown in the following figure. Since, (V) is vertical then the horizontal components of F_1 and F_2 must vanish.



$\sum F_x = 0$ gives

$-F_1 \cos(45^o) + F_2 \cos(45^o) = 0$,

Hence, $F_1 = F_2 = F$

Moreover, $\sum F_y = (V) = 2\, F \sin(45^o)$

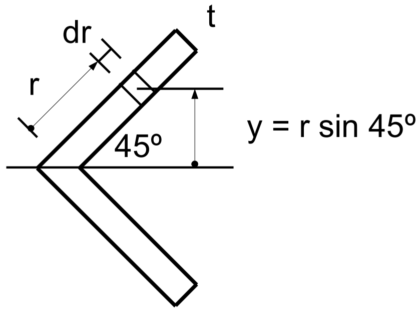
$F = V/\sqrt{2}$

$\sum M$ (about the corner of the median line) = (V) e = 0, since each force F passes through this point.

Therefore, e = 0 and the shear center is at the intersection of the median line of the upper and lower legs. Note that this result is valid even for unequal-leg angle cross-sections.

Determination of the shear stress distribution.

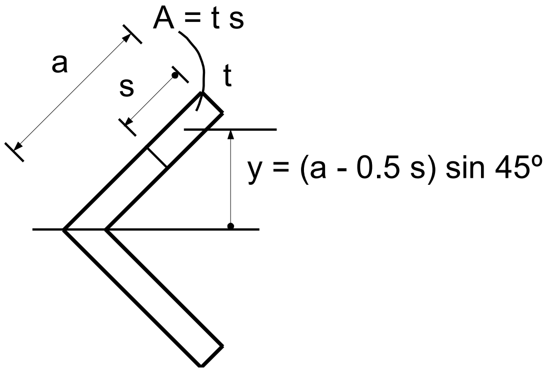
Moment of inertia $I_z = \frac{1}{3}\, t\, a^3$ as is shown in the figure below.



$$I_z = 2\int y^2 dA = 2\int_0^a \left(r\sin\frac{\pi}{4}\right)^2 t\,dr = \frac{t\,a^3}{3}$$

The first moment of area of an area t s (refer to the figure below) is

Q_z = (t s) (a - 0.5 x s) sin(45º)



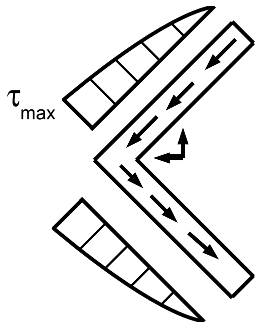
Hence,

$$\tau = \frac{3}{\sqrt{2}\,t\,a^3}\,s\left(a - \frac{s}{2}\right)V$$

The maximum shear stress is at s = a,

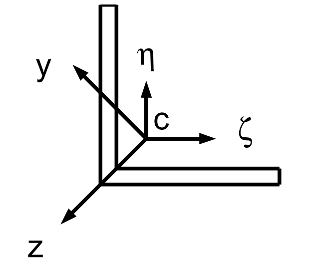
$$\tau_{max} = \frac{3\sqrt{2}}{4}\,\frac{V}{t\,a}$$

The following figure sketches the direction and magnitude of the shear stress along the upper and lower legs.



7.3.1 Comments

(1) The angle a x a x t has $I_z = t\,a^3 / 3$ and $I_y \approx t\,a^3 / 12$, with $I_z / I_y = 4$. A simple way to calculate I_y is by using the formula $I_y = I_\zeta + I_\eta - I_z$, where $I_\zeta = I_\eta = (5/24)\,t\,a^3$.

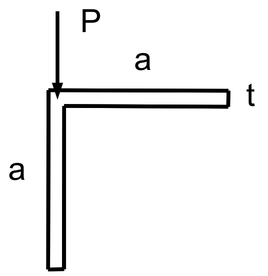


$I_z = t\,a^3/3$ & $I_y = t\,a^3/12$

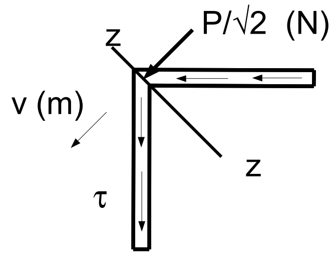
$I_z + I_y = I_\zeta + I_\eta$

$I_z / I_y = 4$

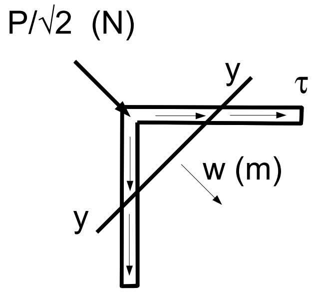
(2) Figure (a) shows a force P acting on a cantilever with length L. Figures (b) and (c) show the shear stress distribution for each component of P parallel to the principal centroidal axes y and z and passing through the shear center. The shear stress in figure (c) is calculated using I_y. The shear stress distribution equals to the sum of the shear stresses of figures (b) and (c).



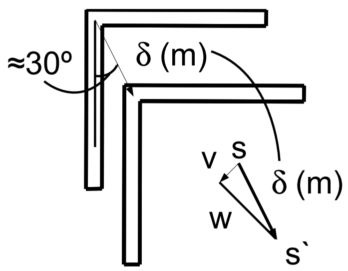
(a)



(b)



(c)



(d) The deflection δ is the resultant of v & w

(3) The deflection of the beam is shown in figure (d). It is not in the direction of P. The basic equation of deflection is u = (F L^3) / (3 E I), where I is a principal centroidal moment of inertia. The deflection v and w of figures (b) and (c) are

v = (P L^3) / (3√2 E I_z) and

w = (P L^3) / (3√2 E I_y)

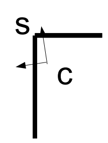
Hence,

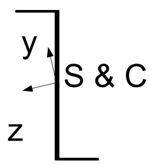
$$\frac{w}{v}=\frac{I_z}{I_y}=4$$

Figure (d) shows how to get the deflection. Try to get the angle of inclination and the magnitude of the deflection δ. This deflection response emphasizes the importance of dealing with the principal directions. Note most finite element programs have three dimensional beam elements that model this out of plane response.

8 Cross- Sections without Symmetry

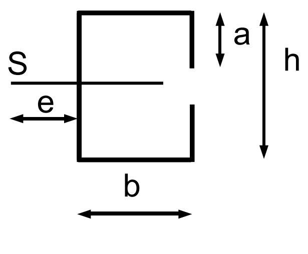
This case is beyond the scope of this course. However, the shear center of unequal-leg angle and a Z-section are easy to derive. The locations of their shear centers are given below:



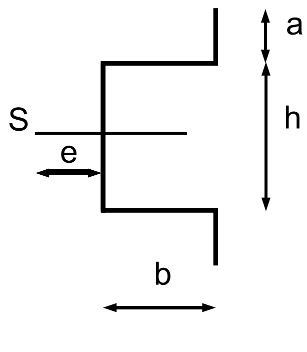


9 Tabulated Locations of the Shear Center

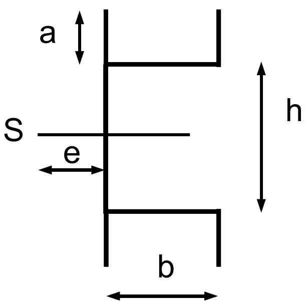
The position of the shear center of the following cross-sections is calculated by e =A/B. Where A and B are listed below. The cross-sections are represented by their median lines. We can get "e" for a channel section by letting a = 0 for cases (1), (2), or (3). (The problem set of the textbook by Gere and Timoshenko contains a large number of cases.)



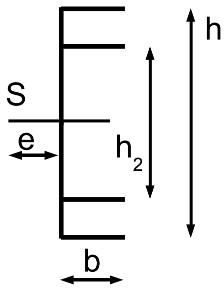
case (1)



case (2)



case (3)



case (4)

Case number	A	B
(1)	$3bh^2(b+2h) - 8ba^3$	$h^2(h+6b+6a) + 4a^2(2a-3h)$
(2)	$3bh^2(b+2h) - 8ba^3$	$h^2(h+6b+6a) + 4a^2(2a+3h)$
(3)	$3bh^2(b+2h) - 8ba^3$	$h^2(h+6b) + 4a(3h^2+4a^2+6ah)$
(4)	$3b^2(h^2+h_2^2)$	$h^3 + 6b(h^2+h_2^2)$

10 References

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